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Charged Spherical Shell in the Field of an Abelian Monopole

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The gravitational field of a charged spherical shell in the presence of a central Abelian monopole is calculated. The nonlinearity in the field is represented by Born–Infeld Lagrangian that contains both invariants of the electromagnetic field. Such a configuration models a gravitationally bound shell of matter and charge that may provide a source of gamma-ray bursts now being found in extragalatic radiation.

1. INTRODUCTION

The past few years have brought a renewed interest in the problem of electromagnetism at extremely high fields (Adler, 1971). The discovery of the 1.8-MeV (e^+e^-) peaks in heavy-ion collisions have prodded theorists to suggest that a new phase of QED exists that may or may not have a confining structure (Galdi, 1989). The past research in nonlinear electrodynamics was divided into two separate approaches; the first was inspired by the Euler–Heisenberg Lagrangian, which represented an effective Lagrangian after the virtual fermions were integrated out of the theory (Euler and Heisenberg, 1936), while the other approach introduced phenomenological Lagrangians of the Born–Infeld type which strove to generate a completely finite electromagnetic field (Born and Infeld, 1934). The research in gauge theory since the introduction of the Higgs field as a symmetry-breaking agent has demonstrated that gauge-Higgs configurations of fields exist that have a topological structure that can be interpreted as monopoles or dyons (Prasad and Sommerfield, 1975). Monopole or dyon solutions exist whenever a group G is broken to a subgroup with a surviving $U(1)$ factor. The far field of a monopole or dyon can be represented as Abelian, wherein the dyon or monopole core contains the non-Abelian

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structure which dies off in the far field. The physical interest in monopoles arose when Rubakov and Callan demonstrated that they could catalyze proton decay with strong interaction rates (Callan, 1982). The present wealth of data on extragalatic radiation reveals gamma-ray bursts that may have an origin in highly nonlinear electromagnetic effects (Ressell and Turner, 1989). The presently accepted theory is that gamma-ray bursts arise from processes within a neutron star, but other processes, such as electromagnetic collapse or black hole radiation, may also contribute to these phenomena. Motivated by the recent interest in the new phase of QED suggested by the 1.8-MeV (e^+e^-) peaks as well as the gamma-ray burst phenomena observed in extragalatic radiation, I study in this note the properties of a configuration of charge governed by nonlinear electromagnetic field with an Abelian monopole at its center. By matching at the boundaries of a spherical shell I calculate the electric and magnetic fields of the configuration as well as the gravitational field inside of and within the shell and well as the fields outside of the shell. The matching of the $\binom{1}{1}$ component of the metric generates an expression for the mass of the shell, while the matching of the $\binom{4}{4}$ component of the metric generates a constraint on the allowed masses and charge of the monopole and charged shell. This simplified model is meant to serve as a starting point for any more complex models of charged matter in the presence of a monopole held together by its own gravitation.

2. CHARGED SHELL IN THE FIELD OF A MONOPOLE

We begin by writing down the following Born–Infeld-type electromagnetic Lagrangian:

$$\mathscr{L} = -\frac{b}{8\pi}\left[\left(1+\frac{J}{b}-\frac{I^2}{16b^2}\right)^{1/2}-1\right]\sqrt{-g} - J_\mu A^\mu \sqrt{-g} \qquad (2.1)$$

Here

$$J = F_{\mu\nu}F^{\mu\nu}, \qquad I = \frac{\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}F_{\mu\nu}}{\sqrt{-g}}, \qquad F_{\mu\nu} = \frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\mu}$$

We station a magnetic charge of magnitude Q at $r=0$ with mass M_x. For the region $0 < r < r_1$, we have no matter other than the dyon and thus $J^\mu = 0$ and

$$\frac{\partial}{\partial x^\nu}(\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}) = 0$$

From the existence of a potential, this gives, with $F_{23} = r^2 \sin\Theta\, B_r$,

$$\frac{\partial}{\partial r}(r^2 B_r) = 0; \qquad B_r = \frac{q}{r^2} \qquad (2.2)$$

For the metric we have

$$(ds)^2 = e^{\nu}(dx^4)^2 - e^{\lambda}(dr)^2 - r^2(d\Theta)^2 - r^2\sin^2\Theta\,(d\phi)^2$$

For the energy-momentum tensor we have from equation (2.1)

$$\begin{aligned} T_{\mu\nu} &= \frac{2}{\sqrt{-g}}\frac{\partial L}{\partial g^{\mu\nu}} \\ &= \frac{b}{8\pi}\left[\left(1+\frac{J}{b}-\frac{I^2}{16b^2}-1\right)^{1/2}\right] g_{\mu\nu} \\ &\quad - \frac{b}{8\pi(1+J/b-I^2/16b^2)^{1/2}}\left[\frac{2}{b}F_{\mu\alpha}F^{\alpha}_{\ \nu} - \frac{I\varepsilon^{ab\alpha\beta}F_{\alpha\beta}F_{ab}}{8b^2(-g)^{1/2}(2)}g_{\mu\nu}\right] \end{aligned} \tag{2.3}$$

$$\begin{aligned} T_{\mu\nu} &= \frac{b}{8\pi}\left[\left(1+\frac{J}{b}-\frac{I^2}{16b^2}\right)^{1/2}-1\right] g_{\mu\nu} \\ &\quad - \frac{b}{8\pi(1+J/b-I^2/16b^2)^{1/2}}\left(\frac{2}{b}F_{\mu\nu}F^{\alpha}_{\ \nu} - \frac{I^2}{16b^2}g_{\mu\nu}\right) \end{aligned} \tag{2.4}$$

For a radial electric and magnetic field $F_{14} = E(r)$, $F_{23} = r^2\sin\Theta\, B_r$ we have $T^1_1 = T^4_4$. For $r < r_1$ (charge-free region) we have

$$T^4_4 = \frac{b}{8\pi}\left[\left(1+\frac{2Br^2}{b}\right)^{1/2}-1\right] = T^1_1$$

for $r < r_1$.

For the $(^4_4)$. Einstein equation we have

$$\begin{aligned} \frac{d}{dr}(r\,e^{-\lambda}) &= 1 - \frac{8\pi G}{c^4}r^2\frac{b}{8\pi}\left[\left(1+\frac{2B_r^2}{b}\right)^{1/2}-1\right] \\ r\,e^{-\lambda} &= r - \frac{2G\bar{M}_x}{c^2} - \frac{8\pi G}{c^4}\frac{b}{8\pi}\int^r r^2\left[\left(1+\frac{2q^2}{br^4}\right)^{1/2}-1\right]dr \end{aligned} \tag{2.5}$$

For $q/r^2 \ll 1$ we have, using $\nu + \lambda = 0$, which follows from $T^1_1 = T^4_4$ and the Einstein equations, upon approximating,

$$\left(1+\frac{2q^2}{br^4}\right)^{1/2} \simeq 1 + \frac{q^2}{br^4}$$

the result

$$e^{\nu} = e^{-\lambda} = 1 - \frac{2GM_x}{rc^2} + \frac{Gq^2}{r^2c^4} \tag{2.6}$$

In general

$$e^{-\lambda} = e^{\nu} = 1 - \frac{2G\bar{M}_x}{rc^2} - \frac{Gb}{c^4 r}\int^r r^2\left[\left(1+\frac{2q^2}{br^4}\right)^{1/2} - 1\right] dr \qquad (2.7)$$

for $r < r_1$.

The reason we have differentiated between the two masses in equations (2.6) and (2.7) lies in the fact that for $q/r^2 \ll 1$, $r < r_1$, we can identify M_x as the mass of the monopole including its electromagnetic contribution because of the form analogous to the Reissner-Nordström solution.

However, the integral in equation (2.7) will give terms that have a $1/r$ dependence that will add to $\bar{M}_x$ to give M_x (the total mass). We next go to the region $r_1 < r < r_2$. In the presence of electric charge, equation (2.1) gives upon variation

$$\frac{\partial}{\partial x^{\nu}}\left[\frac{\sqrt{-g}\,F^{\mu\nu}}{4\pi(1+J/b-I^2/16b^2)^{1/2}} - \frac{I\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}}{32\pi b(1+J/b-I^2/16b^2)^{1/2}}\right] = \sqrt{-g}\,J^{\mu} \qquad (2.8)$$

We choose $J^4 = \rho_0\, dx^4/ds = \rho_0\, e^{-\nu/2}$ for $r_1 < r < r_2$. Equation (2.8) becomes

$$\frac{\partial}{\partial r}\left[\frac{r^2 E(r)}{4\pi(1+2B_r^2/b-2E^2/b-4E^2B_r^2/b^2)^{1/2}} + \frac{(EB_r)B_r r^2}{2\pi b(1+2B_r^2/b-2E^2/b-4E^2B_r^2/b^2)^{1/2}}\right] = \rho_0 r^2 \qquad (2.9)$$

Here we have approximated $e^{\nu} \simeq e^{\lambda} \simeq 1$ in the evaluation of I and the expression for $\sqrt{-g}$, $F^{\mu\nu}$ to allow for an approximate calculation of E. Also, since equation (2.2) holds for $r_1 < r < r_2$, we have

$$\frac{r^2 E}{4\pi(1+2q^2/br^4-2E^2/b-4E^2q^2/b^2r^4)^{1/2}} + \frac{Eq^2}{2\pi br^2(1+2q^2/br^4-2E^2/b-4E^2q^2/b^2r^4)^{1/2}} = \frac{\rho_0 r^3}{3} + C \qquad (2.10)$$

at $r=r_1$, $E=0$; thus, $C=-\rho_0 r_1^3/3$,

$$\left(\frac{1}{4\pi}r^2+\frac{1}{2\pi b}\frac{q^2}{r^2}\right)^2 E^2=\left[\frac{\rho_0}{3}(r^3-r_1^3)\right]^2 \times\left(1+\frac{2q^2}{br^4}-\frac{2E^2}{b}-\frac{4E^2q^2}{b^2r^4}\right) \tag{2.11}$$

Solving for $F_{14}=E(r)$, we have

$$E=F_{14}=\frac{\frac{\rho_0}{3}(r^3-r_1^3)\left(1+\frac{2q^2}{br^4}\right)^{1/2}}{\left[\left(\frac{r^2}{4\pi}+\frac{q^2}{2\pi br^2}\right)^2+\frac{\rho_0(r^3-r_1^3)}{3}\left(\frac{2}{b}+\frac{4q^2}{b^2r^4}\right)\right]^{1/2}} \tag{2.12}$$

for $r_1<r<r_2$.

This is a rather complicated expression for $E(r)$ and any integrals that arise will be left in integral form to avoid any unnecessary complications. For the region $r>r_2$ (outside the charge cloud) we have by integration of equation (2.8) for $J^\mu=0$ and using $\nu+\lambda=0$ for $r>r_2$,

$$\frac{1}{4\pi}\frac{r^2E}{(1+2B_r^2/b-2E^2/b-4E^2B_r^2/b^2)^{1/2}} +\frac{1}{2\pi b}\frac{EB_r^2r^2}{(1+2B_r^2/b-2E^2/b-4E^2B_r^2/b^2)^{1/2}} =\frac{e}{4\pi} \tag{2.13}$$

where e is the total electric charge of the charge cloud. Also, $B=q/r^2$ for $r>r_2$, since the charge cloud does not alter the magnetic charge of the central monopole. Thus, for $r>r_2$

$$F_{14}=E(r)=\frac{e(1+2q^2/br^4)^{1/2}}{[(r^2+2q^2/br^2)^2+2e^2/b+4e^2q^2/b^2r^4]^{1/2}} \tag{2.14}$$

For the matter we have $T_4^4=\varepsilon_0$, $T_2^2=T_3^3=-P$ for $r_1<r<r_2$, where we have assumed vanishing normal pressure for the matter for $r_1<r<r_2$. We now obtain an expression for the metric for $r>r_2$; from equation (2.4) we have

$$T_4^4=T_1^1=\frac{b}{8\pi}\left[\left(1+\frac{2B_r^2}{b}-\frac{4E^2B_r^2}{b^2}-\frac{2E^2}{b}\right)^{1/2}-1\right] -\frac{b}{8\pi(1+2B_r^2/b)}\left[-\frac{2E^2}{b}-\frac{4E^2B_r^2}{b^2}\right]^{1/2} \tag{2.15}$$

for $r>r_2$.

For $r > r_2$ we have the $(^4_4)$ component of the Einstein equation

$$\frac{d}{dr}(r\,e^{-\lambda}) = 1 - \frac{8\pi G}{c^4} r^2 T^4_4 \tag{2.16}$$

where T^4_4 is found from equation (2.15), where only the electromagnetic field contributes. After integration for $r > r_2$ we find

$$e^{\nu} = e^{-\lambda} = 1 - \frac{2G(M_x + M_e)}{rc^2} - \frac{8\pi G}{c^4 r}\int^r r^2 T^4_4\,dr \tag{2.17}$$

The last term in equation (2.17) will give powers of $1/r^2$, $1/r^3, \ldots$, and will not contribute to the mass of the system. Here we have called the constant of integration

$$-\frac{2G}{c^2}(M_x + M_e)$$

where $M_x + M_e$ represents the mass of the monopole plus the mass of the charged shell. We also have for $r > r_2$, $T^1_1 = T^4_4$ implying $\lambda + \nu = 0$ or $e^{-\lambda} = e^{\nu}$ for $r > r_2$ from the $(^1_1)$ and $(^4_4)$ components of the Einstein equations. Thus, in equation (2.17) we have $e^{-\lambda} = e^{\nu}$ for $r > r_2$.

The last term in equation (2.17), as mentioned, will give powers of $1/r^n$ with $n \geq 2$ and we are allowed to interpret the second term as the total mass of the monopole plus charged cloud. For the metric for $r < r_1$ we will use equation (2.6) assuming that $q/r^2 \ll 1$ to justify the expansion in equation (2.5). For $r_1 < r < r_2$ we have for the total energy-momentum tensor

$$\begin{aligned} T^4_4 = \varepsilon_0 &+ \frac{b}{8\pi}\left[\left(1 + \frac{2B_r^2}{b} - \frac{2E^2}{b} - \frac{4E^2B_r^2}{b^2}\right)^{1/2} - 1\right] \\ &- \frac{b}{8\pi}\left[\frac{1}{\sqrt{1 + 2B_r^2/b - 2E^2/b - 4E^2B_r^2/b^2}}\right]\left(-\frac{2E^2}{b} - \frac{4E^2B_r^2}{b^2}\right) \end{aligned} \tag{2.18}$$

$$\begin{aligned} T^1_1 = &\frac{b}{8\pi}\left[\left(1 + \frac{2B_r^2}{b} - \frac{2E^2}{b} - \frac{4E^2B_r^2}{b^2}\right)^{1/2} - 1\right] \\ &- \frac{b}{8\pi}\frac{-2E^2/b - 4E^2B_r^2/b^2}{(1 + 2B_r^2/b - 2E^2/b - 4E^2B_r^2/b^2)^{1/2}} \end{aligned} \tag{2.19}$$

$$\begin{aligned} T^2_2 = T^3_3 = -P &+ \frac{b}{8\pi}\left[\left(1 + \frac{2B_r^2}{b} - \frac{2E^2}{b} - \frac{4E^2B_r^2}{b^2}\right)^{1/2} - 1\right] \\ &- \frac{b}{8\pi(1 + 2B_r^2/b - 2E^2/b - 4E^2B_r^2/b^2)^{1/2}}\left(\frac{2B_r^2}{b} - \frac{4E^2B_r^2}{b^2}\right) \end{aligned} \tag{2.20}$$

Here we have set the normal pressure ($T_1^1 = 0$) equal to zero. We also have approximated $e^{\nu} \simeq e^{\lambda} \simeq 1$ in the expression for the electromagnetic energy-momentum tensor. To solve for M_e, we take the $\binom{4}{4}$ component of the Einstein equations for $r_1 < r < r_2$,

$$\frac{d}{dr}(r\,e^{-\lambda}) = 1 - \frac{8\pi G}{c^4} r^2 T_4^4 \tag{2.21}$$

Integrating from r_1 to r_2, we have

$$(r\,e^{-\lambda})_{r_2} - (r\,e^{-\lambda})_{r_1} = r_2 - r_1 - \frac{8\pi G}{c^4}\int_{r_1}^{r_2} r^2 T_4^4\,dr \tag{2.22}$$

Here we use $(e^{-\lambda})_{r_2}$ from equation (2.17) at $r = r_2$ and $(e^{-\lambda})_{r_1}$ from equation (2.6) at $r = r_1$ and T_4^4 from equation (2.18) for the integrand in equation (2.22). This will allow us to solve for the M_e (the mass of the charged shell). To solve for e^{ν} for $r_1 < r < r_2$ we have for the $\binom{1}{1}$ component of the Einstein equations

$$R_1^1 - \tfrac{1}{2} R\delta_1^1 = -\frac{8\pi G}{c^4} T_1^1$$

$$\frac{e^{-\lambda} - 1}{r^2} + \frac{e^{-\lambda}\nu'}{r} = -\frac{8\pi G}{c^4}\left[\frac{b}{8\pi}\left(1 + \frac{2B_r^2}{b} - \frac{2E^2}{b} - \frac{4E^2B_r^2}{b^2}\right)^{1/2} - \frac{b}{8\pi}\frac{(-2E^2/b - 4E^2B^2/b^2)}{(1 + 2B_r^2/b - 2E^2/b - 4E^2B_r^2/b^2)^{1/2}}\right]$$

or

$$\nu' = \frac{e^{\lambda} - 1}{r} - \frac{8\pi G}{c^4} r\,e^{\lambda}[\cdots]$$

or

$$\nu = \int^r \frac{e^{\lambda} - 1}{r}\,dr - \frac{8\pi G}{c^4}\int^r r\,e^{\lambda}\left[\frac{b}{8\pi}\left(1 + \frac{2B_r^2}{b} - \frac{2E^2}{b} - \frac{4E^2B_r^2}{b^2}\right)^{1/2} - \frac{b}{8\pi}\frac{(-2E^2/b - 4E^2B_r^2/b^2)}{(1 + 2B_r^2/b - 2E^2/b - 4E^2B_r^2/b^2)^{1/2}}\right]dr + C \tag{2.23}$$

From equation (2.21) we have

$$(r\,e^{-\lambda})_r - (r\,e^{-\lambda})_{r_1} = (r - r_1) - \frac{8\pi G}{c^4}\int_{r_1}^{r} r^2 T_4^4\,dr \tag{2.24}$$

Here we insert $(e^{-\lambda})_{r_1}$ from equation (2.6) and use equation (2.18) for T_4^4 for the domain $r_1 < r < r_2$; this gives us $e^{-\lambda}$ for $r_1 < r < r_2$.

Matching equation (2.23) to equation (2.6) for v at $r = r_1$, we have

$$\ln_e\left(1 - \frac{2GM_x}{r_1c^2} + \frac{Gq^2}{r_1^2c^4}\right) = \int^{r_1} \frac{e^\lambda - 1}{r}\,dr - \frac{8\pi G}{c^4}\int^{r_1} r\,e^\lambda \left[\frac{b}{8\pi}\left(1 + \frac{2B_r^2}{b} - \frac{2E^2}{b} - \frac{4E^2B_r^2}{b^2}\right)^{1/2} - \frac{b}{8\pi}\frac{(-2E^2/b - 4E^2B_r^2/b^2)}{(1 + 2B_r^2/b - 2E^2/b - 4E^2B_r^2/b^2)^{1/2}}\right] dr + C \tag{2.25}$$

which determines the constant C in equation (2.23).

In equation (2.25) we use E in equation (2.12), $B_r = q/r^2$, and $e^{-\lambda}$ from equation (2.24) for the expressions for E, b_r, $e^{-\lambda}$ for $r_1 < r < r_2$.

By matching equation (2.23) to equation (2.17) at $r = r_2$ we arrive at a constraint relating M_e, q, e, M_x.

Our next task is to find an approximate solution for M_e. In both regions $r_1 < r < r_2$ and $r > r_2$ we will retain terms up to the cubic terms in the electric and magnetic charge as well as the electric charge density; we will retain terms up to the quartic power in the fields in the energy-momentum tensor.

Equation (2.12) gives to third order for $r_1 < r < r_2$

$$F_{14} = E(r) = \frac{4\pi\rho_0}{3}\left(r - \frac{r_1^3}{r^2}\right)\left(1 - \frac{q^2}{br^4}\right) - \frac{32\pi^3\rho_0^2}{9}\left(r - \frac{r_1^3}{r^2}\right)\left(\frac{1}{r} - \frac{r_1^3}{r^4}\right)\frac{2}{b} + \frac{3}{8}\left(\frac{4\pi\rho_0}{3}\right)\left(\frac{16\pi^2\rho_0}{3}\right)^2\left(\frac{2}{b}\right)^2\left(\frac{1}{r} - \frac{r_1^3}{r^4}\right)^2\left(r - \frac{r_1^3}{r^2}\right) \tag{2.26}$$

And from equation (2.14) for $r > r_2$

$$E \simeq \frac{e}{r^2}\left(1 - \frac{e^2}{br^4} - \frac{q^2}{br^4}\right) \tag{2.27}$$

For the energy-momentum tensor we have from equation (2.15) for $r > r_2$

$$T_4^4 \simeq T_1^1 \simeq \frac{E^2 + B_r^2}{8\pi} + \frac{3}{16\pi b}E^4 - \frac{B_r^4}{16\pi b} + \frac{3}{8\pi b}E^2B_r^2 \tag{2.28}$$

For $r_1 < r < r_2$ we have from equation (2.18)

$$T_4^4 \simeq \frac{E^2 + B_r^2}{8\pi} + \frac{3}{16\pi b} E^4 - \frac{B_r^4}{16\pi b} + \frac{3}{8\pi b} E^2 B_r^2 + \varepsilon_0 \tag{2.29}$$

To find an approximate solution for M_e, we solve for M_e from equation (2.22) using equation (2.17) at $r = r_2$ for $(e^{-\lambda})_{r_2}$, and equation (2.6) for $(e^{-\lambda})_{r_1}$. We also substitute T_4^4 for $r_2 > r > r_1$ from equation (2.29) using equation (2.26) for E and $B = q/r^2$; we also evaluate equation (2.17) for $e^{-\lambda}$ for $r > r_2$ using T_4^4 from equation (2.28) and E from equation (2.14) and $B = q/r^2$. This procedure would give an expression for M_e to fourth order in the electric charge e and the magnetic charge q. The constraint relating M_e, e, q, and M_e would be found by matching equation (2.23) with the known C with equation (2.17) for e^{ν} at $r = r_2$.

3. CONCLUSION

The above procedure has offered us a complete solution to the problem of a charged shell in the presence of an Abelian monopole. The model Lagrangian used in equation (2.1) is intended to simulate the nonlinearity in the field. It is hoped, as mentioned in (Galdi, 1989) that a careful study of the gamma-ray bursts from astrophysical objects might reveal the presence of nonlinear electromagnetic effects in the stellar-like configurations. Possible probes of this nonlinearity in addition to the intense gamma-ray bursts would be anomalous red shifts dependent on q, e as well as mass parameter studies that indicate the possible dependence of the mass of a series of these objects on the electric and magnetic charge of the monopole center and the electrically charged shell.

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